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OPTIMAL STABILIZATION OF LINEAR STOCHASTIC SYSTEMS

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The problem of optimal stabilization is solved for controlled linear systems with white noise. The optimal solution is obtained by the method of successive approximations each of which represents the optimal solution of the related determinate problem. Necessary and sufficient conditions of stabilizability are given.

1. Formulation and transformation of the problem. Consider a stochastic controlled system of the form

$$\frac{dx}{dt} = \left(A + \sum_{r=1}^{k} \sigma_r \xi_r^{\cdot}\right) x + \left(b + \sum_{r=1}^{m} \varphi_r \eta_r^{\cdot}\right) u \tag{1.1}$$

Here x is an n-dimensional phase coordinate vector, u is the scalar control, A and σ_r are constant ($n \times n$)-matrices, b and φ_r are constant n-vectors, $\xi_r^{\cdot}(t)$ $(r = 1, \ldots, k)$ denote the noise present in the object and $\eta_r^{\cdot}(t)$ $(r = 1, \ldots, m)$ is the noise present in the control channel. In addition, all $\xi_r(t)$ and $\eta_r(t)$ are standard Wiener processes independent within the set.

Let us consider a problem of optimal stabilization [1 — 5] of the system (1.1) with the quality criterion

$$I(u) = M \int_0^\infty [x^* G x + \lambda u^2]\, dt, \quad \lambda > 0 \tag{1.2}$$

where G is a positive definite ($n \times n$)-matrix ($G > 0$).

If the Bellman function associated with the problem (1.1), (1.2) is sought as a positive definite quadratic form $x^* M x$, the matrix $M > 0$ satisfies the equation

$$A^* M + MA + \sum_{r=1}^{k} \sigma_r^* M \sigma_r - \frac{Mbb^*M}{\lambda + \varphi(M)} = -G \tag{1.3}$$

$$\varphi(M) = \sum_{r=1}^{m} \varphi_r^* M \varphi_r$$

and the optimal control is given by the formula

$$u_0(x) = -b^* Mx / (\lambda + \varphi(M)) \tag{1.4}$$

Following [3, 4], we perform the following change of variables in (1.3):

$$D = M / (\lambda + \varphi(M)) \tag{1.5}$$

This yields the system

$$\begin{cases} A^*D + DA + \sum_{r=1}^{k} \sigma_r^* D \sigma_r - Dbb^*D = -\mu/\lambda G \\ 1 - \varphi(D) = \mu \end{cases} \tag{1.6}$$

Theorem 1. For the control

$$u_0(x) = -b^*Dx \tag{1.7}$$

to be optimal in the problem (1.1), (1.2) it is necessary and sufficient that system (1.6) has a solution $D > 0$, $\mu > 0$. Such solution of (1.6) is unique.

Proof. Necessity. The existence of a solution of the optimal problem (1.1), (1.2) implies the existence of the solution $M > 0$ of (1.3), and the validity of the formula (1.4) for the optimal control u_0. From (1.5) we find

$$\varphi(D) = \frac{\varphi(M)}{\lambda + \varphi(M)} < 1, \quad \varphi(M) = \frac{\lambda\varphi(D)}{1 - \varphi(D)} \tag{1.8}$$

and the necessity follows at once from (1.5) and (1.6).

Sufficiency. Let $D > 0$, $\mu > 0$ be a solution of the system (1.6). Then $\varphi(D) < 1$. Assuming (see (1.5) and (1.8))

$$M = \lambda D/(1 - \varphi(D)) > 0$$

we find, that M satisfies (1.3) and

$$u_0(x) = -b^*Dx = -b^*Mx/(\lambda + \varphi(M))$$

which implies that the control (1.7) is optimal.

The uniqueness of the solution $D > 0$, $\mu > 0$ of (1.6) follows from the optimality of (1.7) just shown, and this completes the proof of Theorem 1.

2. Necessary and sufficient conditions of stabilizability. The necessary condition for the system (1.1) with noise in the control channel to be stabilizable is, that the corresponding system without noise in the control channel: $\varphi_r = 0$ $(r = 1, \ldots, m)$ is also stabilizable. At the same time, the stabilizability is equivalent to the existence of a solution of the optimal problem (1.1), (1.2) for any positive definite matrix G. It follows therefore from Theorem 1 that for the stabilizability of the system (1.1) with $\varphi_r = 0$ $(r = 1, \ldots, m)$, it is necessary and sufficient that for any $\nu > 0$ there exists a unique solution $D > 0$ of the equation

$$A^*D + DA - Dbb^*D = -\nu G - \sum_{r=1}^{k} \sigma_r^* D \sigma_r \tag{2.1}$$

Theorem 2. The necessary and sufficient condition of stabilizability in the mean square of system (1.1) is the existence of $\nu > 0$ such that the inequality $\varphi(D) < 1$ is satisfied for the solution $D > 0$ of system (2.1).

Proof. The necessity follows at once from the existence of the solution $D > 0$, $\mu > 0$ of the system (1.6) for any G, provided that the system (1.1) is stabilizable.

Sufficiency. Let $D > 0$ be a solution of (2.1), and $\varphi(D) < 1$. We set $\mu = 1 - \varphi(D) > 0$. Then the system (1.6) with $\lambda = \mu/\nu$ has a solution $D > 0$, $\mu > 0$, and consequently the system (1.1) is stabilizable. The stabilizing control can be found from (1.7). This completes the proof of Theorem 2.

We shall show a simple sufficient condition of stabilizability of the system (1.1) without noise in the control channel, i.e. with $\varphi_r = 0$ $(r = 1, \ldots, m)$. This condition will also be a sufficient condition of existence of the solution $D > 0$ of the system (2.1) for any $\nu > 0$.

Theorem 3. If the determinate system

$$dx/dt = Ax + bu \tag{2.2}$$

is fully controllable, system (1.1) with $\varphi_r = 0$ $(r = 1, \ldots, m)$ is stabilizable in the mean square irrespective of any noise in the object.

Without carrying out a detailed proof, we note that by virtue of the complete controllability of system (2.2) its spectrum can be shifted as far to the left as required by choosing an appropriate control. Therefore a stabilizing control can always be selected irrespective of the a priori specified noise in the object.

3. Limiting form of the necessary and sufficient conditions of stabilizability. We shall assume that the system (1.1) with $\varphi_r = 0\,(r = 1, \ldots, m)$, i.e. without noise in the control channel, is stabilizable. Then, as we said in Sect. 2, Eq. (2.1) has a solution $D_\nu > 0$ for any $\nu > 0$. It can be shown that D_ν decreases monotonously as $\nu \to 0$, i.e. $D_{\nu 2} - D_{\nu 1} > 0$ for $\nu_2 > \nu_1$. Therefore the following limit exists:

$$D_0 = \lim_{\nu \searrow 0} D_\nu,\ D_0 \geqslant 0$$

i.e. the matrix D_0 is nonnegative definite.

Theorem 4. The fulfillment of the inequality $\varphi(D_0) < 1$ is a necessary and sufficient condition of stabilizability in the mean square of system (1.1).

The proof follows from Theorem 2 and the fact that the function $\varphi(D_\nu)$ increases monotonously in ν.

From Theorem 4 we can obtain results concerning the stabilizability for the case discussed in [3, 4]. Indeed, we have this case when

$$\sum_{r=1}^{k} \sigma_r^* D \sigma_r = \varphi(D)\, Q \tag{3.1}$$

where Q is a nonnegative definite matrix. Let us consider the equation

$$A^*D + DA - Dbb^*D = -\nu G - Q, \quad \nu > 0 \tag{3.2}$$

Let $\bar{D}_\nu > 0$ be a solution of this equation (we naturally assume that the determinate system (2.2) is stabilizable).

As before, we have the limit

$$\bar{D}_0 = \lim_{\nu \searrow 0} \bar{D}_\nu,\ \bar{D}_0 \geqslant 0$$

Theorem 5. Let the noise in system (1.1) be such that (3.1) is satisfied. Then the necessary and sufficient condition for system (1.1) to be stabilizable in the mean square is that $\varphi(\bar{D}_0) < 1$.

Proof. Necessity. Stabilizability in the mean square of system (1.1) implies, by virtue of Theorem 4, that $\varphi(D_0) < 1$. Since the function $\varphi(D_\nu)$ is monotonous in ν, we can find $\nu = \nu_0$ such that $\varphi(D_{\nu 0}) = 1$. Recalling that D_ν is a solution of (2.1) and $\bar{D}_\nu$ is a solution of (3.2), we find that $D_{\nu_0} = \bar{D}_{\nu_0}$. Using now the fact that the function $\varphi(\bar{D}_\nu)$ is monotonous, we obtain $\varphi(\bar{D}_0) < 1$.

The sufficiency can be proved by repeating the above arguments in the reverse order. This completes the proof of Theorem 5.

When $\nu = 0$, Eq. (3.2), generally speaking, has several nonnegative definite solutions and $\bar{D}_0$ is the largest of them. If however $Q > 0$, then $\bar{D}_0$ coincides with the unique positive definite solution of (3.2) with $\nu = 0$.

It is clear that this solution is optimal for the corresponding determinate problem. We note that in [3, 4] the necessary and sufficient conditions of stabilizability are obtained in the form resembling one obtained here, but only for the case $Q > 0$.

4. Method of successive approximations. We shall assume that the determinate system (2.2) is stabilizable. Then a matrix $D^{(0)} > 0$ can be found satisfying the equation

$$A^*D^{(0)} + D^{(0)}A - D^{(0)}bb^*D^{(0)} = -\nu G, \quad \nu > 0 \tag{4.1}$$

The subsequent approximations $D^{(s)} > 0$ $(s = 1, 2, \ldots)$ will be found from the following recurrent relations:

$$A^*D^{(s)} + D^{(s)}A - D^{(s)}bb^*D^{(s)} = -\nu G - \sum_{r=1}^{k} \sigma_r^*D^{(s-1)}\sigma_r \tag{4.2}$$

Obviously, $D^{(s)} \geqslant D^{(s-1)}$.

Theorem 6. For system (1.1) without noise in the control channel ($\varphi_r = 0$, $r = 1, \ldots, m$) to be stabilizable it is necessary and sufficient that the sequence $D^{(s)}$ has the limit D_ν for any $\nu > 0$.

Proof. Necessity. If the system (1.1) with $\varphi_r = 0$ $(r = 1, \ldots, m)$ is stabilizable, then Eq. (2.1) has a unique positive definite solution $D_\nu > 0$. Clearly, in this case all $D^{(s)} \leqslant D_\nu$ and the limit $\lim_{s\to\infty} D^{(s)} \leqslant D_\nu$ exists. Since this limit also satisfies (2.1), it must therefore be equal to D_ν.

The sufficiency follows at once from the fact that if the limit $\lim_{s\to\infty} D^{(s)}$ exists, it represents a positive definite solution of Eq. (2.1).

We note that the process of finding matrix $D^{(s)} > 0$ from (4.2) is equivalent to solving the problem of analytic construction of regulators for the determinate system (2.2). Therefore, since a program for analytic construction of regulators is available, the recurrent method (4.2) becomes easy to use.

Let us now assume that the system (1.1) with $\varphi_r = 0$ $(r = 1, \ldots, m)$ is stabilizable. Then for any $\nu > 0$ we can use the method (4.2) to obtain the solution $D_\nu > 0$ of Eq. (2.1). Having found the limit $D_0 = \lim_{\nu\searrow 0} D_\nu$, we can use Theorem 4 to settle the problem of stabilizability of (1.1) depending on whether the inequality $\varphi(D_0) < 1$ does or does not hold.

Assuming that the system (1.1) is stabilizable, we finally turn our attention to the question of solving the problem of optimal stabilization of (1.1), (1.2). The solution of this problem can be found according to Theorem 1, if a solution $D_{\mu/\lambda}$ of Eq. (2.1) can be found such that

$$1 - \varphi(D_{\mu/\lambda}) = \mu \tag{4.3}$$

Let us introduce function

$$f(\mu) = 1 - \mu - \varphi(D_{\mu/\lambda}), \quad 0 < \mu < 1$$

It can be proved that the function $f(\mu)$ has a unique root in the interval (0, 1), decreases monotonously and is convex. Therefore the root of (4.3) can be found approximately using e.g. the method of secants or the Newton's method.

Let μ° denote the approximate value of the root of (4.3). Then the matrix $D_{\mu^\circ/\lambda}$ will enable us to construct an approximate optimal solution.

5. Example. Consider the system

$$x_1^{\cdot} = 0.1\,\xi_1^{\cdot}\,x_1 + (1 + 0.2\,\xi_2^{\cdot})\,x_2 \qquad (5.1)$$
$$x_2^{\cdot} = (-1 + 0.2\,\xi_1^{\cdot})\,x_1 + (1 + \varphi\eta^{\cdot})\,u$$

The corresponding determinate system

$$x_1^{\cdot} = x_2, \quad x_2^{\cdot} = -x_1 + u$$

is fully controllable, therefore for any $G > 0, \nu > 0$ Eq. (2.1) has a unique solution $D > 0$.

We note that the equation

$$A^* D + DA - Dbb^* D = -C$$

where C is an arbitrary positive definite matrix, can easily be solved in the present case. Let us write the formulas for the elements of the matrix D $d_{12} = -1 + \sqrt{1 + c_{11}}$, $d_{22} = \sqrt{c_{22} + 2d_{12}}$, $d_{11} = -c_{12} + (1 + d_{12})\,d_{22}$. Using these formulas we can easily carry out the iterative process (4.2) find D_ν for any $\nu > 0$, and then obtain D_0.

In the present case, the elements of the matrix D_0 are, with the accuracy of up to $0.5 \cdot 10^{-5}$, as follows: $d_{11}^\circ = 0.09129$, $d_{12}^\circ = 0.00232$, $d_{22}^\circ = 0.09108$.

In accordance with Theorem 4, we obtain the condition of stabilizability of the system (5.1) in the form $\varphi(D_0) = \varphi^2 \cdot d_{22}^\circ < 1$ or $|\varphi| < 1/\sqrt{d_{22}^\circ} \approx 3.314$.

Following now Sect. 4, we find the optimal control in the problem of minimizing the functional

$$I(u) = M \int_0^\infty (x_1^2 + x_2^2 + u^2)\,dt \qquad (5.2)$$

using the system (5.1) with $\varphi = 2$. We find that the optimal control $u_0(x)$ in the problem (5.1), (5.2) and the optimal value of the functional are as follows:

$$u_0(x) = -0.01572\,x_1 - 0.24531 x_2, \quad x^* M x = 13.280 x_1^2 + 1.676 x_1 x_2 + 13.075 x_2^2.$$

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